

DEGREE 3 ALGEBRAIC MINIMAL SURFACES IN THE 3-SPHERE

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ABSTRACT. We give a local analytic characterization that a minimal surface in the 3-sphere $S^3 \subset \mathbb{R}^4$ defined by an irreducible cubic polynomial is one of the Lawson's minimal tori. This provides an alternative proof of the result by Perdomo (*Characterization of order 3 algebraic immersed minimal surfaces of S^3* , *Geom. Dedicata* 129 (2007), 23–34).

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1. INTRODUCTION

Let $\mathbb{E} = \mathbb{R}^4$ be the four dimensional Euclidean vector space. Let $S^3 \subset \mathbb{E}$ be the unit sphere equipped with the induced Riemannian metric. A *minimal surface* $M \hookrightarrow S^3$ is by definition an immersed surface with vanishing mean curvature, which locally gives an extremal for the area functional. The differential equation for minimal surfaces is an elliptic Monge-Ampere equation defined on the unit tangent bundle of S^3 . It is well known that the minimal surface equation is locally equivalent to the elliptic sinh-Gordon equation for one scalar function of two variables;

$$(1.1) \quad \Delta u + \sinh u = 0.$$

Given a surface $M \hookrightarrow S^3$, consider the cone $O * M \hookrightarrow \mathbb{E}$ over the origin $O \in \mathbb{E}$. Then $M \hookrightarrow S^3$ is minimal whenever $O * M \hookrightarrow \mathbb{E}$ is minimal as a hypersurface in the Euclidean space \mathbb{E} . In this respect, a minimal surface $M \hookrightarrow S^3$ is called *algebraic of degree m* if there exists a nonzero irreducible degree m homogeneous polynomial $F : \mathbb{E} \rightarrow \mathbb{R}$ which vanishes on M . A nonzero irreducible homogeneous polynomial F defines a (possibly singular) minimal surface when

$$(1.2) \quad |\nabla F|^2 \Delta F - (\nabla F)^t \cdot H(F) \cdot (\nabla F) \equiv 0, \quad \text{mod } F.$$

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Here ∇F , ΔF , $H(F)$ denote the gradient, the Laplacian, and the Hessian of the function F respectively with respect to the Euclidean metric of \mathbb{E} , [Hs, p260].

Hsiang classified the homogeneous minimal hypersurfaces in the standard Euclidean spheres, [Hs]. In [Hs], he also showed that the totally geodesic 2-sphere and Clifford torus are the only algebraic minimal surfaces in S^3 of degree ≤ 2 . Lawson constructed an infinite sequence of algebraic minimal tori (or Klein bottles) in S^3 of arbitrary high degree, [La]. Hsiang and Lawson gave an analysis for the minimal submanifolds of low cohomogeneity in general Riemannian homogeneous spaces, where, in particular, the Lawson's sequence of algebraic minimal tori are extended to a countable family of cohomogeneity 1 minimal tori in S^3 , [HsL, p32].

Recently, Perdomo gave a characterization of degree 3 algebraic minimal surfaces in S^3 as one of the Lawson's algebraic minimal tori, [Pe1, Pe2]. One of the main idea of his analysis is that such a minimal surface necessarily contains a great circle, which puts the defining cubic polynomial into a special normal form. The minimal surface equation (1.2) is then applied to successively normalize the polynomial coefficients by differential algebraic analysis.

The purpose of the present paper is to give an analytic characterization of degree 3 algebraic minimal surfaces in S^3 . We employ the method of local differential analysis and show that Lawson's algebraic minimal torus of degree 3 is the only minimal surface in S^3 which satisfies the compatibility equation to lie in the zero locus of an irreducible cubic polynomial.

Main results.

1. The structure equation for the degree 3 algebraic minimal surfaces in S^3 is determined, Theorem 4.1. The analysis for the structure equation provides the formula for the defining cubic polynomial, and it is explicitly identified as the Lawson's degree 3 example.

2. The structure equation shows that the degree 3 algebraic minimal surface is the conjugate surface of a principally bi-planar¹ minimal surface. It has the Killing nullity 1, and the curvature takes values in the closed interval $[-3, \frac{3}{4}]$.

Let us give an outline of the analysis. The problem of finding an algebraic minimal surface in S^3 can be interpreted as the problem of finding a solution to the sinh-Gordon equation (1.1) with the property that it admits an associated, in a certain algebraic manner, polynomial F which satisfies (1.2). This is equivalent to finding a constant section of an appropriate tensor bundle over the minimal surface which vanishes when evaluated on the surface. As the degree of F increases, this problem imposes a sequence of compatibility equations on the higher order structure functions of the minimal surface. For the degree 3 case treated in this paper, the compatibility equations are reduced to a pair of third order equations. **Main results** are obtained by applying the over-determined PDE analysis to these equations.

The present work can be considered as a coordinate free and equivariant interpretation of the aforementioned Perdomo's original characterization. Compared to his analysis, one possible advantage would be that this interpretation of an algebraic minimal surface is susceptible to local analytic method, and could be, in theory, applied to general higher degree cases. However, it is practically difficult to perform the required computations manually. Our analysis relies essentially on the use of computer machine. On the other hand, our analysis also suggests as a byproduct a distinguished class of minimal surfaces defined by an additional fourth order equation ($\Delta_4 = 0$ in (4.8)). An analysis of this class of minimal surfaces will be reported in the subsequent paper.

We carried out the analysis for the degree 4 algebraic minimal surfaces by the same method used in this paper.² The analysis indicates that such a minimal surface is one of the family of cohomogeneity 1 minimal tori described by Hsiang and Lawson. Partly due to the length and complexity of the algebraic analysis involved, we do not present the details of the analysis for degree 4 case in this paper.

¹ A minimal surface in S^3 is *principally bi-planar* when each of its principal curves lies in a totally geodesic $S^2 \subset S^3$, [Ya]. There exist locally one parameter family of distinct principally bi-planar minimal surfaces in S^3 .

² The analysis of the degree 4 algebraic minimal surfaces was the original motivation for the present work. One needs a characterization of the degree 3 surfaces first to exclude them from the analysis for the degree 4 surfaces.

In a different context, similar analysis can be applied to special Legendrian surfaces in the 5-sphere, which are the links of special Lagrangian 3-folds in \mathbb{C}^3 , [Ha]. A special Legendrian surface is called *quasi-algebraic of degree m* if it lies in the zero locus of a nonzero irreducible degree m real homogeneous polynomial $F : \mathbb{C}^3 \rightarrow \mathbb{R}$. The analysis shows that;

1. a quasi-algebraic degree 2 special Legendrian surface is necessarily one of the cohomogeneity 1 surfaces treated by Haskins, [Ha],
2. a quasi-algebraic degree 3 special Legendrian surface with an appropriate \mathbb{Z}_3 symmetry is necessarily a quasi-algebraic surface of degree 2.

The computation indicates in fact that any quasi-algebraic degree 3 special Legendrian surface is necessarily a quasi-algebraic surface of degree 2. We do not have a complete proof of this claim at this time of writing.

The paper is organized as follows. In Section 2, we set the basic structure equation for the minimal surfaces in S^3 , and compute its infinite sequence of prolongations, (2.8), Lemma 2.14. In order to simplify the computations, the induced complex structure on the minimal surface is utilized, and the structure equations are written in complex form. In Section 3, we give a description of the method to detect the compatibility equations for a minimal surface to be algebraic. The method is applied to two simple cases of degree 1, and degree 2 algebraic minimal surfaces, Theorem 3.6, Theorem 3.8. In Section 4, we present the results of differential analysis for degree 3 algebraic minimal surfaces, Theorem 4.1. After a preliminary reduction in Section 4.1, the analysis is divided into two cases. The differential analysis shows that a degree 3 algebraic minimal surface necessarily satisfies an auxiliary equation either of order 3, Section 4.2, or of order 4, Section 4.3. The auxiliary equation of order 3 is compatible and supports a defining irreducible cubic polynomial, which is explicitly verified as the one given by Lawson.

The majority of computations were performed using the computer algebra system **Maple** with **diffforms** package.³

Throughout the paper, a surface is a connected smooth two dimensional manifold.

2. MINIMAL SURFACES IN S^3

In this section, we set the basic structure equations for a minimal surface in S^3 . In Section 2.1, we apply the moving frame method to determine the structure equation for an immersed oriented minimal surface in S^3 . In Section 2.2, we compute the infinite prolongation of the structure equation explicitly, Lemma 2.14.

The analysis and results in this section are classical, and well known. For the standard reference on the theory of minimal surfaces in S^3 , we refer to [La] and the references therein.

The basic structure equations established in this section will be used implicitly throughout the paper.

2.1. Structure equation. Let $\mathbb{E} = \mathbb{R}^4$ be the four dimensional Euclidean vector space. Let $S^3 \subset \mathbb{E}$ be the unit sphere equipped with the induced Riemannian metric. The special orthogonal group SO_4 acts transitively on S^3 as a group of isometry, and $S^3 = SO_4/SO_3$.

Let $\Lambda \rightarrow S^3$ be the S^2 -bundle of unit tangent vectors. Let $Gr^+(2, \mathbb{E})$ be the Grassmannian of two dimensional oriented subspaces of \mathbb{E} . SO_4 acts transitively on both Λ and $Gr^+(2, \mathbb{E})$, and there exists the incidence double fibration;

³ The **Maple** worksheet is available upon request by email.

$$\begin{array}{ccc}
& \text{SO}_4 & \\
& \downarrow \pi & \\
& \Lambda = \text{SO}_4/\text{SO}_2 & \\
\pi_0 \swarrow & & \searrow \pi_1 \\
\text{S}^3 & & \text{Gr}^+(2, \mathbb{E}) = \text{SO}_4/(\text{SO}_2 \times \text{SO}_2)
\end{array}$$

Figure 2.1. Double fibration

To fix the notation once and for all, let us define the projection maps π , π_0 , and π_1 explicitly. Let $e = (e_0, e_1, e_2, e_3)$ denote the $\text{SO}_4 \subset \text{GL}_4\mathbb{R}$ frame of \mathbb{E} . Define

$$\begin{aligned}
(2.1) \quad \pi(e) &= (e_0, e_0 \wedge e_3), \\
\pi_0(e_0, e_0 \wedge e_3) &= e_0, \\
\pi_1(e_0, e_0 \wedge e_3) &= e_0 \wedge e_3.
\end{aligned}$$

The SO_4 -frame e satisfies the structure equation

$$\begin{aligned}
(2.2) \quad de_A &= \sum_B e_B \omega_A^B, \\
\omega_B^A + \omega_A^B &= 0,
\end{aligned}$$

for the Maurer-Cartan form (ω_B^A) of SO_4 . (ω_B^A) satisfies the compatibility equation

$$(2.3) \quad d\omega_B^A + \sum_C \omega_C^A \wedge \omega_B^C = 0.$$

Let $x : M \hookrightarrow \text{S}^3$ be an immersed oriented surface. By the general theory of moving frames, there exists a lift $\tilde{x} : M \hookrightarrow \Lambda$ such that $e_0 = x$, and e_3 is the oriented normal to the surface x . Let $\tilde{x}^*\text{SO}_4 \rightarrow M$ be the pulled back SO_2 -bundle. We continue to use (ω_B^A) to denote the pulled back Maurer-Cartan form on $\tilde{x}^*\text{SO}_4$. From (2.1), (2.2), the initial state of (ω_B^A) on $\tilde{x}^*\text{SO}_4$ takes the form

$$(2.4) \quad (\omega_B^A) = \begin{pmatrix} \cdot & -\omega^1 & -\omega^2 & \cdot \\ \omega^1 & \cdot & \omega_2^1 & \omega_3^1 \\ \omega^2 & \omega_1^2 & \cdot & \omega_3^2 \\ \cdot & \omega_1^3 & \omega_2^3 & \cdot \end{pmatrix}.$$

Here \cdot denotes zero, and $\omega_0^A = \omega^A$, $A = 1, 2$. By definition, $I = \langle de_0, de_0 \rangle = (\omega^1)^2 + (\omega^2)^2$ is the induced Riemannian metric of the immersed surface, where $\langle \cdot, \cdot \rangle$ is the inner product of \mathbb{E} .

Differentiating $\omega_0^3 = 0$, one gets

$$\omega_1^3 \wedge \omega^1 + \omega_2^3 \wedge \omega^2 = 0.$$

By Cartan's lemma, there exist coefficients h_{AB} , $A, B = 1, 2$, symmetric in indices such that

$$\omega_A^3 = \sum_B h_{AB} \omega^B.$$

The structure equation shows that the quadratic differential

$$(2.5) \quad \text{II} = \omega_A^3 \circ \omega^A = h_{AB} \omega^A \circ \omega^B$$

is well defined on M . II is the second fundamental form of the immersed surface x .

Definition 2.6. Let $x : M \hookrightarrow \text{S}^3$ be an immersed oriented surface. Let II be the quadratic differential (2.5) which is the second fundamental form of the immersed surface x . x is minimal when the trace of the quadratic differential II with respect to the induced metric I vanishes, or equivalently when

$$h_{11} + h_{22} = 0.$$

From now on, a minimal surface would mean an immersed oriented minimal surface.

In order to utilize the induced complex structure on M as a Riemann surface, let us introduce the complexified structure equation. Let $\mathbb{E}^{\mathbb{C}} = \mathbb{E} \otimes \mathbb{C}$ be the complexification, and consider the following $\mathbb{E}^{\mathbb{C}}$ -frame.

$$(2.7) \quad \begin{aligned} e^{\mathbb{C}} &= (e_0, E_1, E_{-1}, e_3), \\ E_1 &= \frac{1}{2}(e_1 - i e_2), \quad i^2 = -1, \\ E_{-1} &= \overline{E_1}. \end{aligned}$$

Rewriting (2.2), (2.4) with respect to $e^{\mathbb{C}}$, one gets

$$(2.8) \quad de^{\mathbb{C}} = e^{\mathbb{C}} \begin{pmatrix} \cdot & -\frac{1}{2}\overline{\omega} & -\frac{1}{2}\omega & \cdot \\ \omega & -i\rho & \cdot & -\overline{h_2}\overline{\omega} \\ \overline{\omega} & \cdot & i\rho & -h_2\omega \\ \cdot & \frac{1}{2}h_2\omega & \frac{1}{2}\overline{h_2}\overline{\omega} & \cdot \end{pmatrix},$$

where we set

$$\begin{aligned} \omega &= \omega^1 + i\omega^2, \\ \rho &= \omega_2^1, \\ h_2 &= h_{11} - i h_{12}. \end{aligned}$$

Differentiating (2.8), one gets the compatibility equations

$$(2.9) \quad \begin{aligned} d\omega &= i\rho \wedge \omega, \\ d\rho &= K \frac{i}{2} \omega \wedge \overline{\omega}, \quad \text{where } K = 1 - h_2 \overline{h_2}, \\ dh_2 + 2i h_2 \rho &\equiv 0, \quad \text{mod } \omega. \end{aligned}$$

Here K is the curvature of the induced metric on the minimal surface.

Remark 2.10. *The subscript '2' in the notation ' h_2 ' represents the weight of the action by the structure group SO_2 . It is convenient for a computational purpose.*

(2.9) shows that the quadratic differential

$$(2.11) \quad \Pi^{\mathbb{C}} = h_2 \omega \circ \omega$$

is well defined on M , and is holomorphic with respect to the complex structure on M defined by the $(1,0)$ -form ω . $\Pi^{\mathbb{C}}$ is the complexified second fundamental form of the minimal surface.

Definition 2.12. *Let $x : M \hookrightarrow S^3$ be an immersed oriented minimal surface. The holomorphic quadratic differential $\Pi^{\mathbb{C}}$, (2.11), is the Hopf differential of the minimal surface. The zero set of $\Pi^{\mathbb{C}}$ is the umbilic divisor.*

Example 2.13. *Consider an immersed minimal sphere $S^2 \hookrightarrow S^3$. Since S^2 supports no nonzero holomorphic differentials of positive degree, $\Pi^{\mathbb{C}} = 0$ and the structure coefficient h_2 vanishes identically. A minimal sphere in S^3 is necessarily totally geodesic.*

Conversely, it is clear from the structure equation (2.8) that a minimal surface in S^3 with $h_2 \equiv 0$ is locally equivalent to the totally geodesic sphere.

For a compact minimal surface $M \hookrightarrow S^3$ of genus $g \geq 1$, the umbilic divisor has degree $4g - 4$ by Riemann-Roch theorem. In particular, the umbilic divisor of a minimal torus is empty.

2.2. Prolongation. In this section, we compute the infinite sequence of prolongations of the structure equation (2.9). The prolonged structure equation will be used implicitly for the differential analysis in Section 4.

Let us introduce the higher order derivatives of the structure coefficient h_2 in (2.9) inductively by

$$dh_j + i j h_j \rho = h_{j+1} \omega + h_{j,-1} \bar{\omega}, \quad j = 2, 3, \dots$$

For a notational purpose, denote $h_{-j} = \bar{h}_j$, $j = 2, 3, \dots$. For instance, the curvature of the surface is written as $K = 1 - h_2 \bar{h}_2 = 1 - h_2 h_{-2}$.

Lemma 2.14.

$$\begin{aligned} h_{2,-1} &= 0, \\ h_{j+1,-1} &= \sum_{s=0}^{j-2} c_{js} h_{j-s} \partial^s K, \quad \text{for } j \geq 2, \quad \text{where} \\ c_{js} &= \frac{(j+s+2)}{2} \frac{(j-1)!}{(j-s-2)!(s+2)!} = \frac{(j+s+2)}{2j} \binom{j}{s+2}. \end{aligned}$$

Here by definition $\partial^s K = \delta_{0s} - h_{2+s} h_{-2}$ for $s \geq 0$.

Proof. Differentiating dh_j and collecting $\omega \wedge \bar{\omega}$ -terms, one gets

$$h_{j+1,-1} = \partial h_{j,-1} + \frac{j}{2} h_j K,$$

where $\partial h_{j,-1}$ is the ω -component of $dh_{j,-1}$ ($\partial^s K$ is defined similarly). The formula is verified by direct computation. Note that

$$\begin{aligned} a_{j0} &= \frac{(j+2)(j-1)}{4}, \\ a_{j(j-2)} &= 1. \quad \square \end{aligned}$$

Example 2.15. Consider $\mathbb{R}^4 = \mathbb{C} \oplus \mathbb{C} = \mathbb{C}^2$. Clifford torus is the minimal surface in S^3 defined as the product of circles

$$\{(z_1, z_2) \in \mathbb{C}^2 \mid |z_1|^2 = |z_2|^2 = \frac{1}{2}\}.$$

The induced Riemannian metric is flat, and the curvature K vanishes. Lemma 2.14 shows that

$$0 = \partial K = -h_3 h_{-2}.$$

Clifford torus is not totally geodesic, and this implies $h_3 \equiv 0$.

Conversely, consider a minimal surface in S^3 such that the structure function $h_3 \equiv 0$. By Lemma 2.14, $\bar{\partial} h_3 = h_2 K \equiv 0$, and either $h_2 \equiv 0$ (and $K = 1$), or $K = 0$. In the latter case, it is well known that the surface is locally congruent to Clifford torus.

3. ALGEBRAIC MINIMAL SURFACES

Let $S^m(\mathbb{E})$ be the vector space of real, homogeneous degree m polynomials on the four dimensional Euclidean vector space \mathbb{E} . By the metric duality, we identify $S^m(\mathbb{E})$ with the space of symmetric m -tensors $Sym^m(\mathbb{E})$. Let $S(\mathbb{E}) = \bigoplus_{m=0}^{\infty} S^m(\mathbb{E})$.

Definition 3.1. Let $x : M \hookrightarrow S^3 \subset \mathbb{E}$ be a minimal surface in the unit sphere. x is algebraic if there exists a nonzero homogeneous polynomial $F \in S(\mathbb{E})$ which vanishes on x . For each $m \geq 0$, let $\mathcal{J}_x^m \subset S^m(\mathbb{E})$ be the subspace of degree m polynomials vanishing on x . Degree of an algebraic minimal surface x is the minimum integer m such that \mathcal{J}_x^m is nontrivial.

Note by definition that for an algebraic minimal surface x of degree m_0 , \mathcal{J}_x^m is nontrivial for all $m \geq m_0$.

In this section, we first describe the idea of how the over-determined PDE analysis can be applied to detect the compatibility equations for a minimal surface to be algebraic, Section 3.1. We then apply this to the cases of degree 1, and degree 2 algebraic minimal surfaces, Section 3.2, Section 3.3.

We continue to use the structure equations established in Section 2. The method of local differential analysis described in this section will be applied to degree 3 algebraic minimal surfaces in Section 4.

3.1. Structure equation. Let $x : M \hookrightarrow S^3$ be a minimal surface with the associated complexified frame $e^{\mathbb{C}}$, (2.7). Consider the trivial bundle $S^m(\mathbb{E}) \times S^3 \rightarrow S^3$, and the induced bundle $x^*(S^m(\mathbb{E}) \times S^3) \rightarrow M$. A section of $x^*(S^m(\mathbb{E}) \times S^3)$ over M can be represented in terms of the frame $e^{\mathbb{C}}$ as follows.

$$F = \sum_{i,j,k,l \geq 0}^{i+j+k+l=m} p_{ijkl} e_0^i e_3^j E_1^k E_{-1}^l, \quad p_{ijkl} = \overline{p_{ijlk}},$$

for a set of coefficients p_{ijkl} . Equivalently, such F is an $S^m(\mathbb{E})$ -valued function on M .

The condition that F is a constant section, or $F \in S^m(\mathbb{E})$ with an abuse of notation, is expressed by

$$(3.2) \quad dF = 0.$$

Expanding the exterior derivative dF by Leibniz rule in the variables $\{e_0, e_3, E_1, E_{-1}\}$, one gets the structure equation for the coefficients $\{p_{ijkl}\}$;

$$(3.3) \quad dp_{ijkl} = \sum \phi_{ijkl}^{i'j'k'l'} p_{i'j'k'l'},$$

where $\phi_{ijkl}^{i'j'k'l'}$ is determined as a linear combination of the components of the Maurer-Cartan form (2.8).

Remark 3.4. The formulae for $\{\phi_{ijkl}^{i'j'k'l'}\}$ will be given explicitly in Section 3.2, Section 3.3, Section 4 for the cases $m = 1, 2, 3$ respectively.

The condition that the polynomial F vanishes on x is expressed by

$$(3.5) \quad p_{m000} = 0.$$

The idea is then to apply the structure equation (3.3) repeatedly starting from the initial state (3.5) by using (2.9) and (2.14). As one differentiates, the successive derivatives of (3.5) imply more and more linear compatibility equations on the polynomial coefficients $\{p_{ijkl}\}$, thereby reducing the number of independent coefficients.

In the case F is irreducible, the reduction process eventually leads to a single independent polynomial coefficient, and the integrability equations for a minimal surface to support such an irreducible polynomial are expressed as a set of algebraic equations on the structure functions $\{h_{\pm 2}, h_{\pm 3}, \dots\}$. For the minimal surface satisfying these integrability equations, the formula for F is obtained by evaluating the polynomial coefficients at an appropriate generic point of the minimal surface.

In the next two sections, we examine the well known cases of degree 1, and degree 2 minimal surfaces following this method just described. These two cases not only serve as exercise for the differential analysis of the more complicated case of degree 3 surfaces in Section 4, but they are also necessary prerequisites. From the remark below Definition 3.1, one needs the characterization of degree 1, and degree 2 algebraic minimal surfaces to exclude them from the analysis for the degree 3 surfaces.

3.2. Degree 1 algebraic minimal surfaces. In this section, we give a characterization of degree 1 algebraic minimal surfaces.

Theorem 3.6. Let $x : M \hookrightarrow S^3 \subset \mathbb{E}$ be an algebraic minimal surface of degree 1. Then the structure functions satisfy

$$h_{\pm 2} = 0.$$

The minimal surface is congruent to a part of the totally geodesic sphere.

Proof of theorem is presented below in 3 steps.

Let us write a real, homogeneous degree 1 polynomial $F \in S^1(\mathbb{E})$ in terms of the frame $e^{\mathbb{C}}$.

$$F = p_0 e_0 + p_1 E_1 + p_{-1} E_{-1} + p_3 e_3 \in S^1(\mathbb{E}).$$

The equation $dF = 0$ is equivalent to the following set of structure equations for the polynomial coefficients.

$$(3.7) \quad \begin{aligned} dp_0 - p_{-1}\omega - p_1\bar{\omega} &= 0, \\ dp_3 + p_1 h_2 \omega + p_{-1} h_{-2} \bar{\omega} &= 0, \\ dp_1 - i p_1 \rho + \frac{1}{2} p_0 \omega - \frac{1}{2} p_3 h_{-2} \bar{\omega} &= 0, \\ dp_{-1} + i p_{-1} \rho + \frac{1}{2} p_0 \bar{\omega} - \frac{1}{2} p_3 h_2 \omega &= 0. \end{aligned}$$

Differentiating the coefficients of F would mean applying this structure equation from now on.

Step 1. Assume the initial condition,

$$p_0 = 0.$$

By the metric duality, this is equivalent to that F vanishes on the minimal surface x .

Step 2. Differentiating $p_0 = 0$, one gets

$$p_{\pm 1} = 0.$$

At this stage, the polynomial is reduced to $F = p_3 e_3$.

Step 3. Differentiating $p_{\pm 1} = 0$, one gets

$$p_3 h_{\mp 2} = 0.$$

- If the structure functions $h_{\pm 2}$ do not vanish identically, $p_3 = 0$ and the minimal surface does not support a nonzero degree 1 polynomial vanishing on the surface.

- If $h_{\pm 2} = 0$ identically, the minimal surface is congruent to a part of the totally geodesic sphere, Example 2.13. The structure equation (3.7) is reduced to $dp_3 = 0$, which is compatible.

Fix a point x_0 on the minimal surface. Choose an orthonormal coordinate $\{x_0, x_1, x_2, x_3\}$ of \mathbb{E} so that one has the following identification at x_0 by the metric duality.

$$\begin{aligned} e_0 &= x_0, \\ E_{\pm 1} &= \frac{1}{2}(x_1 \mp i x_2), \\ e_3 &= x_3. \end{aligned}$$

Up to scale, one may assume $p_3 = 1$. The degree 1 polynomial F is given by

$$F = x_3. \quad \square$$

3.3. Degree 2 algebraic minimal surfaces. In this section, we give a characterization of degree 2 algebraic minimal surfaces.

Theorem 3.8. *Let $x : M \hookrightarrow S^3 \subset \mathbb{E}$ be an algebraic minimal surface of degree 2. Then the structure functions satisfy*

$$\begin{aligned} 1 - h_2 h_{-2} &= 0, \\ h_{\pm 3} &= 0. \end{aligned}$$

The minimal surface is congruent to a part of Clifford torus. For an orthonormal coordinate $\{x_0, x_1, x_2, x_3\}$ of \mathbb{E} , the surface is defined by the quadratic polynomial

$$F = x_0 x_3 - x_1 x_2.$$

Proof of theorem is presented below in 5 steps.

Let us write a real, homogeneous degree 2 polynomial $F \in S^2(\mathbb{E})$ in terms of the frame $e^{\mathbb{C}}$.

$$\begin{aligned} F = & p_{0,0}e_0^2 + 2p_{0,3}e_0e_3 + p_{3,3}e_3^2 \\ & + 2r_{0,1}e_0E_1 + 2r_{0,-1}e_0E_{-1} + 2r_{3,1}e_3E_1 + 2r_{3,-1}e_3E_{-1} \\ & + 4r_{1,1}E_1^2 + 8r_{1,-1}E_1E_{-1} + 4r_{-1,-1}E_{-1}^2 \in S^2(\mathbb{E}). \end{aligned}$$

We implicitly assume the appropriate conjugation relations among the coefficients so that F is real, i.e., $r_{0,-1} = \overline{r_{0,1}}$, etc. The equation $dF = 0$ is equivalent to the following set of structure equations for the polynomial coefficients.

$$\begin{aligned} (3.9) \quad & dp_{0,0} - r_{0,-1}\omega - r_{0,1}\overline{\omega} = 0, \\ & dp_{0,3} + \frac{1}{2}(-r_{3,-1} + h_2r_{0,1})\omega + \frac{1}{2}(-r_{3,1} + h_{-2}r_{0,-1})\overline{\omega} = 0, \\ & dp_{3,3} + h_2r_{3,1}\omega + h_{-2}r_{3,-1}\overline{\omega} = 0, \\ & dr_{0,1} - i r_{0,1}\rho + (p_{0,0} - 2r_{1,-1})\omega + (-h_{-2}p_{0,3} - 2r_{1,1})\overline{\omega} = 0, \\ & dr_{0,-1} + i r_{0,-1}\rho + (p_{0,0} - 2r_{1,-1})\overline{\omega} + (-h_2p_{0,3} - 2r_{-1,-1})\omega = 0, \\ & dr_{3,1} - i r_{3,1}\rho + (p_{0,3} + 2h_2r_{1,1})\omega + (-h_{-2}p_{3,3} + 2h_{-2}r_{1,-1})\overline{\omega} = 0, \\ & dr_{3,-1} + i r_{3,-1}\rho + (p_{0,3} + 2h_{-2}r_{-1,-1})\overline{\omega} + (-h_2p_{3,3} + 2h_2r_{1,-1})\omega = 0, \\ & dr_{1,1} - 2i r_{1,1}\rho + \frac{1}{2}r_{0,1}\omega - \frac{1}{2}h_{-2}r_{3,1}\overline{\omega} = 0, \\ & dr_{1,-1} + \frac{1}{4}(r_{0,-1} - h_2r_{3,1})\omega + \frac{1}{4}(r_{0,1} - h_{-2}r_{3,-1})\overline{\omega} = 0, \\ & dr_{-1,-1} + 2i r_{-1,-1}\rho + \frac{1}{2}r_{0,-1}\overline{\omega} - \frac{1}{2}h_2r_{3,-1}\omega = 0. \end{aligned}$$

Differentiating the coefficients of F would mean applying this structure equation from now on.

Step 1. Assume the initial condition,

$$Eq_{1,1} : p_{0,0} = 0.$$

By the metric duality, this is equivalent to that the quadratic polynomial F vanishes on the minimal surface x .

Step 2. Differentiating $p_{0,0} = 0$, one gets

$$\begin{aligned} Eq_{2,1} : r_{0,1} &= 0, \\ Eq_{2,2} : r_{0,-1} &= 0. \end{aligned}$$

Step 3. Differentiating $r_{0,\pm 1} = 0$, one gets

$$\begin{aligned} (3.10) \quad & Eq_{3,1} : h_{-2}p_{0,3} + 2r_{1,1} = 0, \\ & Eq_{3,2} : r_{1,-1} = 0, \\ & Eq_{3,3} : h_2p_{0,3} + 2r_{-1,-1} = 0. \end{aligned}$$

One may solve for $\{r_{1,1}, r_{1,-1}, r_{-1,-1}\}$ from these equations.

Step 4. Differentiating (3.10), one gets

$$\begin{aligned} (3.11) \quad & Eq_{4,1} : h_2r_{3,1} = 0, \\ & Eq_{4,2} : h_{-2}r_{3,-1} = 0. \end{aligned}$$

Since the minimal surface has degree 2, $h_{\pm 2}$ do not vanish identically, and this implies $r_{3,\pm 1} = 0$.

Step 5. Differentiating (3.11), one finally gets

$$\begin{aligned} Eq_{5,1} : h_2 p_{3,3} &= 0, \\ Eq_{5,2} : h_{-2} p_{3,3} &= 0, \\ Eq_{5,3} : (1 - h_2 h_{-2}) p_{0,3} &= 0. \end{aligned}$$

As in **Step 4**, $h_{\pm 2}$ does not vanish identically, and $p_{3,3} = 0$. At this stage, the quadratic polynomial F is reduced to $F = 2p_{0,3}(-e_0 e_3 + h_{-2} E_1^2 + h_2 E_{-1}^2)$.

- If the curvature $(1 - h_2 h_{-2})$ does not vanish identically, $p_{0,3} = 0$ and the minimal surface does not support a nonzero degree 2 polynomial vanishing on the surface.

- If $1 - h_2 h_{-2} = 0$ identically, the minimal surface is congruent to Clifford torus, Example 2.15. The structure equation (3.9) is reduced to $dp_{0,3} = 0$, which is compatible. By the existence and uniqueness theorem of ODE, there exists up to scale a unique nonzero quadratic polynomial that vanishes on Clifford torus.

Fix a point x_0 on Clifford torus. From the compatibility equation $1 - h_2 h_{-2} = 0$, one may adapt the frame at x_0 so that $h_{\pm 2} = 1$. Choose an orthonormal coordinate $\{x_0, x_1, x_2, x_3\}$ of \mathbb{E} so that one has the following identification at x_0 by the metric duality.

$$\begin{aligned} e_0 &= x_0, \\ E_1 &= \frac{1}{2}(x_1 - i x_2) \exp(i \frac{\pi}{4}), \\ E_{-1} &= \frac{1}{2}(x_1 + i x_2) \exp(-i \frac{\pi}{4}), \\ e_3 &= x_3. \end{aligned}$$

Up to scale, one may assume $p_{0,3} = -\frac{1}{2}$. The degree 2 polynomial F is given by

$$F = x_0 x_3 - x_1 x_2. \quad \square$$

Comparing the analysis for degree 1, and degree 2 minimal surfaces, it is evident that the analysis for higher degree algebraic minimal surfaces would follow the same path, but that the computational complexity would increase due to possibly the large number of terms involving the higher order structure functions $h_{\pm 2}, h_{\pm 3}, \dots$.⁴ We shall show in the next section that the required differential analysis is manageable for the degree 3 surfaces, and one may recover the Perdomo's result essentially by local analysis.

4. DEGREE 3 ALGEBRAIC MINIMAL SURFACES

In this section, we give a local analytic characterization of degree 3 algebraic minimal surfaces in S^3 .

Theorem 4.1. [Pe2] *Let $x : M \hookrightarrow S^3 \subset \mathbb{E}$ be an algebraic minimal surface of degree 3. Then the structure functions satisfy*

$$\begin{aligned} h_2^3 h_{-3}^2 + h_{-2}^3 h_3^2 &= 0, \\ h_3 h_{-3} + 4 h_2^2 h_{-2}^2 + 4 h_2 h_{-2} &= 10 (h_2 h_{-2})^{\frac{3}{2}}. \end{aligned}$$

For an orthonormal coordinate $\{x_0, x_1, x_2, x_3\}$ of \mathbb{E} , the surface is defined by the cubic polynomial

$$F = -2x_0 x_1 x_2 + x_3(x_1^2 - x_2^2).$$

a) x is the conjugate surface of a principally bi-planar minimal surface. It has the Killing nullity 1, and there exists up to scale a unique Killing vector field of SO_4 that is tangent to x .

b) The curvature of the minimal surface takes values in the closed interval $[-3, \frac{3}{4}]$.

⁴ This does not necessarily mean that the higher degree algebraic minimal surfaces are characterized by a set of higher order equations. But the differential analysis itself does require manipulation of the higher order terms.

The minimal surface defined by the above cubic polynomial is one of the infinite sequence of algebraic minimal tori constructed by Lawson, [La, p350].

Remark 4.2. *Given the degree 3 algebraic minimal torus, one may ask if the conjugate surface, which is principally bi-planar, is also algebraic. An analysis indicates that this conjugate minimal surface does not close up to become a torus.*

Perdomo first gave the characterization of degree 3 algebraic minimal surfaces in S^3 , [Pe2]. One of the main idea of his analysis is that such a minimal surface necessarily contains a great circle, and that the gradient of the defining cubic polynomial vanishes at a point on the great circle. This puts the cubic polynomial into a special normal form. By applying the minimal surface equation (1.2), the characterization is reduced essentially to solving a set of algebraic equations among the constant polynomial coefficients.

The analysis for the degree 3 case proceeds similarly as for the degree 1, and the degree 2 cases treated in the previous section. On the other hand, a cubic polynomial on $\mathbb{E} = \mathbb{R}^4$ has 20 coefficients. The computational complexity increases as degree increases, and one has to take higher order derivatives in order to access the compatibility equations for a minimal surface to be algebraic. For the degree 3 case, the analysis requires differentiating six times.

Proof of Theorem 4.1 consists of two parts. In the first part, a preliminary analysis is done to reduce the number of independent polynomial coefficients from 20 to 6, Section 4.1. After differentiating four times, the analysis divides into two cases depending on whether a certain structure invariant of the minimal surface vanishes or not. In the second part, we carry out the differential analysis for each case under the appropriate assumptions on the structure functions, Section 4.2, Section 4.3.

For the rest of the paper, we assume $h_{\pm 2} \neq 0$, $1 - h_2 h_{-2} \neq 0$, and $h_{\pm 3} \neq 0$ to exclude degree 1, and degree 2 algebraic minimal surfaces.

4.1. Preliminary analysis. Let us write a real, homogeneous degree 3 polynomial $F \in S^3(\mathbb{E})$ in terms of the frame $e^{\mathbb{C}}$, (2.7).

$$\begin{aligned}
 (4.3) \quad F = & p_0 e_0^3 + p_1 e_0^2 e_3 + p_2 e_0 e_3^2 + p_3 e_3^3 \\
 & + 8 q_0 E_1^3 + 8 q_1 E_1^2 E_{-1} + 8 q_2 E_1 E_{-1}^2 + 8 q_3 E_{-1}^3 \\
 & + 2 r_{0,1} e_0^2 E_1 + 2 r_{0,-1} e_0^2 E_{-1} \\
 & + 2 r_{1,1} e_0 e_3 E_1 + 2 r_{1,-1} e_0 e_3 E_{-1} \\
 & + 2 r_{2,1} e_3^2 E_1 + 2 r_{2,-1} e_3^2 E_{-1} \\
 & + 4 r_{0,2} e_0 E_1^2 + 4 r_{0,0} e_0 E_1 E_{-1} + 4 r_{0,-2} e_0 E_{-1}^2 \\
 & + 4 r_{1,2} e_3 E_1^2 + 4 r_{1,0} e_3 E_1 E_{-1} + 4 r_{1,-2} e_3 E_{-1}^2.
 \end{aligned}$$

We implicitly assume the appropriate conjugation relations among the coefficients so that F is real, i.e., $r_{0,-1} = \overline{r_{0,1}}$, etc.

$F \in S^3(\mathbb{E})$ is a constant cubic polynomial. Differentiating (4.3) by Leibniz rule, the equation $dF = 0$ is equivalent to the following set of structure equations for the polynomial coefficients.

$$\begin{aligned}
(4.4) \quad & dp_0 - r_{0,-1}\omega - r_{0,1}\bar{\omega} = 0, \\
& dp_1 + (-r_{1,-1} + h_2 r_{0,1})\omega + (-r_{1,1} + h_{-2} r_{0,-1})\bar{\omega} = 0, \\
& dp_2 + (-r_{2,-1} + h_2 r_{1,1})\omega + (-r_{2,1} + h_{-2} r_{1,-1})\bar{\omega} = 0, \\
& dp_3 + h_2 r_{2,1}\omega + h_{-2} r_{2,-1}\bar{\omega} = 0, \\
& dq_0 - 3i q_0 \rho + \frac{1}{2} r_{0,2}\omega - \frac{1}{2} h_{-2} r_{1,2}\bar{\omega} = 0, \\
& dq_1 - i q_1 \rho + \frac{1}{2} (r_{0,0} - h_2 r_{1,2})\omega + \frac{1}{2} (r_{0,2} - h_{-2} r_{1,0})\bar{\omega} = 0, \\
& dq_2 + i q_2 \rho + \frac{1}{2} (r_{0,-2} - h_2 r_{1,0})\omega + \frac{1}{2} (r_{0,0} - h_{-2} r_{1,-2})\bar{\omega} = 0, \\
& dq_3 + 3i q_3 \rho - \frac{1}{2} h_2 r_{1,-2}\omega + \frac{1}{2} r_{0,-2}\bar{\omega} = 0, \\
& dr_{0,1} - i r_{0,1}\rho + \frac{1}{2} (-2 r_{0,0} + 3 p_0)\omega + \frac{1}{2} (-4 r_{0,2} - h_{-2} p_1)\bar{\omega} = 0, \\
& dr_{0,-1} + i r_{0,-1}\rho + \frac{1}{2} (-4 r_{0,-2} - h_2 p_1)\omega + \frac{1}{2} (-2 r_{0,0} + 3 p_0)\bar{\omega} = 0, \\
& dr_{1,1} - i r_{1,1}\rho + (-r_{1,0} + p_1 + 2 h_2 r_{0,2})\omega + (-2 r_{1,2} - h_{-2} p_2 + h_{-2} r_{0,0})\bar{\omega} = 0, \\
& dr_{1,-1} + i r_{1,-1}\rho + (-2 r_{1,-2} - h_2 p_2 + h_2 r_{0,0})\omega + (-r_{1,0} + p_1 + 2 h_{-2} r_{0,-2})\bar{\omega} = 0, \\
& dr_{2,1} - i r_{2,1}\rho + \frac{1}{2} (p_2 + 4 h_2 r_{1,2})\omega + \frac{1}{2} (2 h_{-2} r_{1,0} - 3 h_{-2} p_3)\bar{\omega} = 0, \\
& dr_{2,-1} + i r_{2,-1}\rho + \frac{1}{2} (2 h_2 r_{1,0} - 3 h_2 p_3)\omega + \frac{1}{2} (p_2 + 4 h_{-2} r_{1,-2})\bar{\omega} = 0, \\
& dr_{0,2} - 2i r_{0,2}\rho + (-q_1 + r_{0,1})\omega + \frac{1}{2} (-6 q_0 - h_{-2} r_{1,1})\bar{\omega} = 0, \\
& dr_{0,0} + \frac{1}{2} (-4 q_2 - h_2 r_{1,1} + 2 r_{0,-1})\omega + \frac{1}{2} (-4 q_1 - h_{-2} r_{1,-1} + 2 r_{0,1})\bar{\omega} = 0, \\
& dr_{0,-2} + 2i r_{0,-2}\rho + \frac{1}{2} (-6 q_3 - h_2 r_{1,-1})\omega + (-q_2 + r_{0,-1})\bar{\omega} = 0, \\
& dr_{1,2} - 2i r_{1,2}\rho + \frac{1}{2} (r_{1,1} + 6 h_2 q_0)\omega + (-h_{-2} r_{2,1} + h_{-2} q_1)\bar{\omega} = 0, \\
& dr_{1,0} + \frac{1}{2} (r_{1,-1} + 4 h_2 q_1 - 2 h_2 r_{2,1})\omega + \frac{1}{2} (r_{1,1} + 4 h_{-2} q_2 - 2 h_{-2} r_{2,-1})\bar{\omega} = 0, \\
& dr_{1,-2} + 2i r_{1,-2}\rho + (-h_2 r_{2,-1} + h_2 q_2)\omega + \frac{1}{2} (r_{1,-1} + 6 h_{-2} q_3)\bar{\omega} = 0.
\end{aligned}$$

Differentiating the coefficients of F would mean applying this structure equation from now on.

The preliminary analysis consists of the following 5 steps. The condition that F vanishes on the minimal surface serves as the initial condition for the over-determined PDE analysis. By successively applying the above structure equation, we reduce the number of independent polynomial coefficients to $20 - 14 = 6$. The reduction process stops at **Step 5**, where the analysis divides into two cases.

Step 1. Assume the initial condition

$$Eq_{1,1} : p_0 = 0.$$

By the metric duality, this equivalent to that the cubic polynomial F vanishes on the minimal surface x .

Step 2. Differentiating $p_0 = 0$, one gets

$$\begin{aligned} Eq_{2,1} : r_{0,1} &= 0, \\ Eq_{2,2} : r_{0,-1} &= 0. \end{aligned}$$

Step 3. Differentiating $r_{0,\pm 1} = 0$, one gets

$$(4.5) \quad \begin{aligned} Eq_{3,1} : \frac{1}{2} h_{-2} p_1 + 2 r_{0,2} &= 0, \\ Eq_{3,2} : r_{0,0} &= 0, \\ Eq_{3,3} : \frac{1}{2} h_2 p_1 + 2 r_{0,-2} &= 0. \end{aligned}$$

One may solve for $\{r_{0,2}, r_{0,0}, r_{0,-2}\}$ from these equations.

Step 4. Differentiating (4.5), one gets

$$(4.6) \quad \begin{aligned} Eq_{4,1} : 2 q_2 + \frac{1}{2} h_2 r_{1,1} &= 0, \\ Eq_{4,2} : 2 q_1 + \frac{1}{2} h_{-2} r_{1,-1} &= 0, \\ Eq_{4,3} : 6 q_0 + \frac{1}{2} p_1 h_{-3} + \frac{3}{2} h_{-2} r_{1,1} &= 0, \\ Eq_{4,4} : 6 q_3 + \frac{1}{2} p_1 h_3 + \frac{3}{2} h_2 r_{1,-1} &= 0. \end{aligned}$$

One may solve for $\{q_0, q_1, q_2, q_3\}$ from these equations.

Step 5. Differentiating (4.6), one gets

$$(4.7) \quad \begin{aligned} Eq_{5,1} : 2 h_3 r_{1,1} + 6 h_2 r_{1,0} + (-h_2 + h_2^2 h_{-2}) p_1 &= 0, \\ Eq_{5,2} : h_{-2} r_{1,-2} + h_2 r_{1,2} + \frac{1}{2} h_2 h_{-2} p_2 &= 0, \\ Eq_{5,3} : 2 h_{-3} r_{1,-1} + 6 h_{-2} r_{1,0} + (-h_{-2} + h_{-2}^2 h_2) p_1 &= 0, \\ Eq_{5,4} : 12 h_{-3} h_2^2 r_{1,0} + 6 h_3 h_{-2}^2 r_{1,-2} - 6 h_{-2} h_3 h_2 r_{1,2} + (-h_3 h_2 h_{-4} - 2 h_{-3} h_2^2 + 2 h_{-3} h_2^3 h_{-2}) p_1 &= 0, \\ Eq_{5,5} : 12 (h_3^2 h_{-2}^3 + h_{-3}^2 h_2^3) r_{1,0} + (-h_3 h_{-2}^2 h_{-3} h_4 - 2 h_3^2 h_{-2}^3 + 2 h_3^2 h_{-2}^4 h_2 - h_{-3} h_2^2 h_3 h_{-4} \\ - 2 h_{-3}^2 h_2^3 + 2 h_{-3}^2 h_2^4 h_{-2}) p_1 &= 0. \end{aligned}$$

Since we are assuming that the algebraic minimal surface has degree 3, both $h_{\pm 2}, h_{\pm 3}$ do not vanish identically, otherwise the algebraic minimal surface would have degree 1, or 2 by Theorem 3.6, Theorem 3.8. One may thus solve for $\{r_{1,1}, p_2, r_{1,-1}, r_{1,2}\}$ from $\{Eq_{5,1}, Eq_{5,2}, Eq_{5,3}, Eq_{5,4}\}$. Note that $Eq_{5,5}$ is equivalent to $\overline{Eq_{5,4}}$ modulo $Eq_{5,4}$. \square

At this step, the analysis is divided into the following two cases. Set

$$(4.8) \quad \begin{aligned} \Delta_3^+ &= h_{-2}^3 h_3^2 + h_2^3 h_{-3}^2, \\ \Delta_4 &= h_2 h_{-3}^2 h_4 - h_{-2} h_3^2 h_{-4}. \end{aligned}$$

• Case $\Delta_3^+ = 0$, Section 4.2: It will be shown that the degree 3 algebraic minimal surface in Theorem 4.1 belongs to this case. The minimal surfaces with $\Delta_3^+ = 0$ are the conjugate surfaces of the *principally bi-planar* minimal surfaces. A minimal surface in S^3 is principally bi-planar when each of its principal curves is planar, [Ya]. The principally bi-planar minimal surfaces are characterized by the equation

$\Delta_3^- = h_{-2}^3 h_3^2 - h_2^3 h_{-3}^2 = 0$.⁵ Both of the Δ_3^\pm -null minimal surfaces belong to the wider class of the cohomogeneity 1 minimal surfaces, [HsL, p32].⁶

• Case $\Delta_3^+ \neq 0$, $\Delta_4 = 0$, Section 4.3: The analysis shows that for a minimal surface with $\Delta_3^+ \neq 0$, the fourth order equation $\Delta_4 = 0$ is a necessary condition to be algebraic of degree 3. But, a further analysis shows that the resulting structure equation is not compatible, and there does not exist any degree 3 algebraic minimal surfaces in this case.

In the following two sections, we present the differential analysis for each case.

4.2. Case $h_{-2}^3 h_3^2 + h_2^3 h_{-3}^2 = 0$. In this section, we show that, up to motion by SO_4 , there exists a unique degree 3 algebraic minimal surface in S^3 that satisfies $\Delta_3^+ = h_{-2}^3 h_3^2 + h_2^3 h_{-3}^2 = 0$.

We first determine the structure equation for the minimal surfaces with $\Delta_3^+ = 0$, Section 4.2.1. A first integral J is defined, (4.10), and it follows that there exists locally a one parameter family of Δ_3^+ -null surfaces. We then continue the analysis of Section 4.1 and show that exactly one of the Δ_3^+ -null surfaces is algebraic of degree 3, Section 4.2.2.

4.2.1. Structure equation. Let $x : M \hookrightarrow S^3$ be a Δ_3^+ -null surface. Differentiating $\Delta_3^+ = 0$, one may solve for $h_{\pm 4}$ and get

$$(4.9) \quad h_4 = \frac{1}{2} \frac{h_2^2 h_{-3}(-2 h_2 h_{-2} + 2 h_2^2 h_{-2}^2 - 3 h_3 h_{-3})}{h_3 h_{-2}^3},$$

$$h_{-4} = \overline{h_4}.$$

Here we assume $h_{\pm 2}, h_{\pm 3}$ are nonzero. A direct computation shows that the structure equation is compatible with this relation, i.e., $d^2 h_{\pm 3} = 0$ is an identity.

Set

$$(4.10) \quad J = \frac{1}{8} \frac{(h_3 h_{-3} + 4 h_2 h_{-2} + 4 h_2^2 h_{-2}^2)}{(h_2 h_{-2})^{\frac{3}{2}}}, \quad J > 0.$$

Then $dJ = 0$, and J is a first integral for the Δ_3^+ -null surfaces.

We claim that $J \in (1, \infty)$, and that h_2 is nowhere zero. Let us denote $|h_2| = a$, $|h_3| = b$. The equation (4.10) is written as

$$(4.11) \quad b^2 + 4 a^2(1 + a^2 - 2 J a) = 0.$$

This implies that $J \geq 1$, and that either $a \equiv 0$, which is excluded, or a takes values in the closed interval $[J - \sqrt{J^2 - 1}, J + \sqrt{J^2 - 1}]$. When $J = 1$, one has that $a \equiv 1$, $b \equiv 0$, which is also excluded.

The structure equations (4.9) and (4.10) will be used implicitly for the analysis in the next subsection.

4.2.2. Differential analysis. We now continue the analysis of Section 4.1. Due to their lengths, the exact expressions for $Eq_{6,1}, Eq_{6,2}, Eq_{6,3}, Eq_{6,4}; Eq_7$ below will be postponed to Appendix.

Step 5'. Assume $\Delta_3^+ = 0$. Then (4.9) implies that $Eq_{5,5} = 0$.

Step 6. Differentiating (4.7) and equating modulo $\Delta_3^+ = 0$, one gets a set of four independent equations $\{Eq_{6,1}, Eq_{6,2}, Eq_{6,3}, Eq_{6,4}\}$ (see Appendix). One may solve for $\{r_{1,0}, r_{2,1}, r_{2,-1}r_{1,-2}\}$ from these equations.

Step 7. Differentiating $Eq_{6,1}$, one gets Eq_7 (see Appendix). One may solve for p_3 from this equation.

At this step, p_1 is the only remaining independent polynomial coefficient, and it satisfies the structure equation, (4.4), of the form

$$dp_1 \equiv 0, \quad \text{mod } p_1.$$

⁵ One may verify this by a straightforward moving frame computation. We omit the details.

⁶ One may verify by direct computation that the cohomogeneity 1 minimal surfaces in S^3 are characterized by the pair of fourth order equations $\Delta_4^+ = 2 h_2 h_{-3}^2 h_4 - h_3 h_{-3}(3 h_3 h_{-3} + 2 h_2 h_{-2} K) = 0$, $\Delta_4^- = \Delta_4^+ = 0$.

From the uniqueness theorem of ODE, if p_1 vanishes at a point of the minimal surface, it vanishes identically, which implies the cubic polynomial F vanishes. We therefore assume p_1 is nowhere zero from now on.

Step 8. Differentiating the remaining equations $\{Eq_{6,2}, Eq_{6,3}, Eq_{6,4}; Eq_7\}$, one gets a single compatibility equation, up to scale by non-identically zero terms;

$$(8J + 10)(8J - 10) = 0.$$

Here J is the first integral (4.10). Since $J > 1$, one must have

$$(4.12) \quad J = \frac{5}{4}.$$

Moreover, a direct computation shows that $d^2p_1 = 0$ is an identity with this relation.

By the existence and uniqueness theorem of ODE, there exists up to scale a unique nonzero cubic polynomial that vanishes on the Δ_3^+ -null minimal surface with the first integral $J = \frac{5}{4}$.

Fix a point x_0 on the minimal surface. Choose an orthonormal coordinate $\{x_0, x_1, x_2, x_3\}$ of \mathbb{E} so that one has the following identification at x_0 by the metric duality.

$$\begin{aligned} e_0 &= x_1, \\ E_1 &= \frac{1}{2}(x_0 - ix_2) \exp(i\frac{\pi}{4}), \\ E_{-1} &= \frac{1}{2}(x_0 + ix_2) \exp(-i\frac{\pi}{4}), \\ e_3 &= x_3. \end{aligned}$$

Up to scale, one may assume $p_1 = \frac{1}{3}$ at x_0 . From the analysis of Section 4.2.1, $|h_2| = a$ takes values in the closed interval $[\frac{1}{2}, 2]$. Evaluating $\lim_{a \rightarrow 2} F$ with $h_2 = a$, the degree 3 polynomial F is given by

$$(4.13) \quad F = -2x_0x_1x_2 + x_3(x_1^2 - x_2^2).$$

4.2.3. Proof of Theorem 4.1.

a) From (4.13), set $z_1 = x_3 + ix_0$, $z_2 = x_1 + ix_2$. Then $F = \operatorname{Re}(z_1 z_2^2)$. It is clear that F is invariant under a subgroup $\operatorname{SO}_2 \subset \operatorname{SO}_4$. It is known that a minimal surface in S^3 with Killing nullity ≥ 2 is either the totally geodesic sphere or Clifford torus, [HsL].

b) By definition of the first integral J in (4.11), $|h_2| = a$ takes values in the closed interval $[\frac{1}{2}, 2]$. The curvature is given by $K = 1 - a^2$. \square

4.3. Case $h_{-2}^3 h_3^2 + h_2^3 h_{-3}^2 \neq 0$. In this section, we show that there does not exist a degree 3 algebraic minimal surface in S^3 with the property that $\Delta_3^+ = h_{-2}^3 h_3^2 + h_2^3 h_{-3}^2$ is not identically zero.

The analysis will show that under the condition $\Delta_3^+ \neq 0$, a degree 3 algebraic minimal surface necessarily satisfies a fourth order equation $\Delta_4 = h_2 h_{-3}^2 h_4 - h_{-2} h_3^2 h_{-4} = 0$. We first determine the structure equation for the minimal surfaces with $\Delta_4 = 0$, Section 4.3.1. We then continue the analysis of Section 4.1 and show that none of the Δ_4 -null surfaces is algebraic of degree 3, Section 4.3.2.

4.3.1. Structure equation. Let $x : M \hookrightarrow S^3$ be a Δ_4 -null surface. Differentiating $\Delta_4 = 0$, one may solve for $h_{\pm 5}$ and get

$$(4.14) \quad h_5 = \frac{5h_{-2}h_3^2h_{-3} - 7h_{-2}^2h_3^2h_{-3}h_2 + 4h_4h_{-4}h_3h_{-2} - 4h_4h_{-3}h_2h_{-2} + 4h_4h_{-3}h_2^2h_{-2}^2 - 2h_4h_{-3}^2h_3}{2h_{-3}^2h_2},$$

$$h_{-5} = \overline{h_5}.$$

Here we assume $h_{\pm 2}, h_{\pm 3}$ are nonzero. A direct computation shows that the structure equation is compatible with this relation, i.e., $d^2h_{\pm 4} = 0$ is an identity.

The structure equation (4.14) will be used implicitly for the analysis in the next subsection.

4.3.2. *Differential analysis.* We now continue the analysis of Section 4.1. Due to their lengths, the exact expressions for $Eq_{6,1}, Eq_{6,2}, Eq_{6,3}; Eq_7$ below will be postponed to Appendix.

Step 5'. Assume $\Delta_3^+ \neq 0$. One may solve for $r_{1,0}$ from $Eq_{5,5}$.

Step 6. Differentiating $Eq_{5,1}, Eq_{5,2}$ from (4.7), one gets a set of three independent equations $\{Eq_{6,1}, Eq_{6,2}, Eq_{6,3}\}$ (see Appendix). One may solve for $\{r_{2,1}, r_{1,-2}, r_{2,-1}\}$ from these equations.

Step 7. Differentiating $Eq_{6,1}$, one gets a single equation Eq_7 (see Appendix). One may solve for p_3 from this equation.

At this step, p_1 is the only remaining independent polynomial coefficient, and it satisfies the structure equation, (4.4), of the form

$$dp_1 \equiv 0, \quad \text{mod } p_1.$$

As in Section 4.2.2, we assume p_1 is nowhere zero from now on.

Step 8. Differentiating $Eq_{5,4}, Eq_{5,5}$ from (4.7), one gets a set of two equations, which allow one to solve for $h_{\pm 5}$ (see Appendix). Note that we have not assumed $\Delta_4 = 0$ yet.

At this step, we observe that the coefficient p_3 is real. Evaluating $p_3 - \overline{p_3} = 0$ with the relations obtained so far, one gets the following compatibility equation;

$$\Delta_4 \Delta_4' = 0,$$

where

$$\begin{aligned} \Delta_4' = & (-4 h_2^4 h_{-3}^2 h_{-2} + 3 h_2^4 h_{-2}^2 h_{-4} + 10 h_2^3 h_{-3}^2 - 3 h_2^3 h_{-2} h_{-4} + 3 h_2^2 h_4 h_{-2}^4 - 3 h_2 h_{-2}^3 h_4 \\ & - 4 h_2 h_{-2}^4 h_3^2 + 10 h_{-2}^3 h_3^2). \end{aligned}$$

A short analysis shows that Δ_4' vanishes only when $h_3 \equiv 0$, which is excluded. Hence a degree 3 algebraic minimal surface with $\Delta_3^+ \neq 0$ must satisfy $\Delta_4 = 0$.

We assume the structure equations (4.14) from now on.

Step 9. The cubic polynomial F , (4.3), is real. Evaluating $F - \overline{F} = 0$, one gets the single compatibility equation

$$(-h_{-2}^3 h_3^2 + h_2^3 h_{-3}^2) \Delta_4'' = 0,$$

where

$$\Delta_4'' = 3 h_2 h_{-3} h_4^2 + (3 h_2^2 h_3 h_{-2}^2 - 3 h_2 h_3 h_{-2} - 4 h_{-3} h_3^2) h_4 - 4 h_{-2}^2 h_2 h_3^3 + 10 h_{-2} h_3^3.$$

A further differential analysis shows that $(-h_{-2}^3 h_3^2 + h_2^3 h_{-3}^2)$ vanishes only when $h_3 \equiv 0$, which is excluded (the differential analysis for this case is a little evolved, but straightforward. We shall omit the details). Hence a degree 3 algebraic minimal surface with $\Delta_3^+ \neq 0$ must also satisfy $\Delta_4'' = 0$.

Step 10. Differentiating $Eq_{6,2}$ modulo $\Delta_4' = 0$, one gets another single compatibility equation, up to scale by non-identically zero terms;

$$Eq_{10,1} : (3 h_2 h_{-2} - 2) h_4 - 4 h_3^2 h_{-2} = 0.$$

Comparing $Eq_{10,1}$ with $\Delta_4'' = 0$, one gets

$$Eq_{10,2} : 39 h_2^2 h_{-2}^2 - 56 h_2 h_{-2} + 20 + 16 h_3 h_{-3} = 0.$$

Differentiating this equation again, one gets

$$Eq_{10,3} : 93 h_2^2 h_{-2}^2 - 122 h_2 h_{-2} + 40 + 32 h_3 h_{-3} = 0.$$

$Eq_{10,2}$ and $Eq_{10,3}$ are compatible only when $h_{\pm 2} = 0$. \square

APPENDIX.

We record the exact formulae of the long expressions omitted in the main text.

A-1. Section 4.2

$$\begin{aligned}
Eq_{6,1} : & (-6h_2^3 + 6h_2^4h_{-2})r_{1,0} - 6h_2^2h_{-2}h_3r_{2,-1} + 4h_3^2h_{-2}r_{1,-2} \\
& + (h_2^5h_{-2}^2 - h_2^3h_{-3}h_3 + h_2^3 - 2h_2^4h_{-2})p_1 = 0, \\
Eq_{6,2} : & (36h_2^6h_{-2}h_{-3}h_3 - 36h_2^5h_{-2}^2h_{-3}h_3)r_{2,1} + (54h_{-3}^3h_2^5h_{-2}h_3 + 72h_2^3h_{-2}^5h_3^3 \\
& - 72h_2^4h_{-2}^6h_3^2 + 36h_{-3}^2h_2^6h_{-2}^2 + 48h_{-3}h_2^2h_3^3h_{-2}^4 - 36h_{-3}^2h_2^7h_{-2}^3)r_{2,-1} \\
& + (-36h_{-3}^3h_2^3h_{-2}h_3^2 - 32h_3^4h_{-2}^4h_{-3} + 24h_2^5h_{-2}^3h_3^2h_3 \\
& - 48h_2h_3^3h_{-2}^5 + 48h_3^3h_{-2}^6h_2^2 - 24h_2^4h_{-2}^3h_3h_{-2}^2)r_{1,-2} \\
& + (-6h_2^8h_{-3}^3h_{-2}^2 + 8h_3^3h_{-2}^3h_2^3h_{-2}^2 + 6h_{-3}^3h_2^7h_{-2} + 10h_2^3h_{-2}^3h_{-3}h_3^2 \\
& + 9h_2^6h_{-3}^4h_3 - 2h_{-2}^5h_2^5h_{-3}h_3^2 - 8h_2^4h_{-2}^4h_{-3}h_3^2)p_1 = 0, \\
Eq_{6,3} : & (-36h_2^4h_{-2}^6h_3^2 + 102h_{-3}^3h_2^5h_{-2}h_3 - 36h_{-2}^2h_2^7h_3^2 \\
& + 36h_3^2h_{-2}^5h_2^3 + 36h_{-3}^2h_2^6h_{-2}^2 + 48h_{-3}h_2^2h_3^3h_{-2}^4)r_{2,-1} \\
& + (24h_3^3h_{-2}^6h_2^2 - 24h_2h_3^3h_{-2}^5 - 48h_2^4h_{-2}^2h_3h_{-2}^2 \\
& + 48h_2^5h_{-2}^3h_3^2h_3 - 68h_{-3}^3h_2^3h_{-2}h_3^2 - 32h_3^4h_{-2}^4h_{-3})r_{1,-2} \\
& + (14h_3^3h_{-2}^3h_2^3h_{-2}^2 - 6h_2^2h_3^3h_{-2}^2h_{-2}^2 - 6h_2^8h_{-3}^3h_{-2}^2 + 6h_2^3h_{-2}^3h_{-3}h_3^2 \\
& + 17h_2^6h_{-3}^4h_3 - 6h_{-2}^5h_2^5h_{-3}h_3^2 + 6h_{-3}^3h_2^7h_{-2})p_1 = 0, \\
Eq_{6,4} : & (-12h_{-3}^2h_3^4h_2^5h_{-2}^2 + 108h_{-3}^4h_2^5h_{-2}^2h_3^2)r_{1,-2} \\
& + (-12h_2^{10}h_{-3}^4h_{-2}^3 + 42h_2^9h_{-3}^4h_{-2}^2 + 34h_2^8h_{-3}^5h_{-2}h_3 - 30h_2^8h_{-2}h_{-3}^4 \\
& - 12h_2^7h_{-2}^6h_{-3}^2h_3^2 - 85h_2^7h_3h_{-3}^5 + 24h_{-2}^2h_2^6h_{-2}^5h_3^2 \\
& + 34h_2^5h_{-3}^3h_3^3h_{-2}^4 - 12h_2^5h_{-2}^3h_3^2h_{-2}^4 + 18h_3^5h_{-2}^7h_{-3}h_2^2 \\
& - 34h_2^4h_{-3}^3h_3^3h_{-2}^2 - 18h_{-2}^8h_3^4h_2^2 - 24h_2^3h_3^4h_{-2}^2h_{-3}^4 - 27h_3^6h_{-2}^5h_{-2}^2 \\
& + 18h_3^4h_{-2}^7h_2^2 + 9h_{-3}h_3^5h_{-2}^6h_2)p_1 = 0, \\
Eq_7 : & (-1458h_2^9h_{-2}^5h_{-3}^4h_3^3 + 162h_2^6h_{-2}^8h_3^5h_{-2}^2)p_3 \\
& + (108h_3^4h_{-2}^{10}h_2^8h_{-3} + 894h_2^9h_{-2}^5h_{-3}^4h_3^3 + 294h_2^6h_{-2}^8h_3^5h_{-2}^2 \\
& - 762h_2^8h_3^3h_{-3}^4h_{-2}^2 + 452h_2^8h_3^4h_{-3}^5h_{-2}^2 - 72h_2^9h_{-2}^5h_{-3}^3h_3^2 + 216h_2^{10}h_{-2}^6h_{-3}^3h_3^2 \\
& - 192h_2^7h_{-2}^9h_3^5h_{-2}^2 + 306h_2^{11}h_{-2}^7h_3^2h_{-2} - 216h_2^{11}h_{-3}^3h_{-2}^7h_3^2 \\
& - 216h_{-2}^9h_3^4h_2^7h_{-3} + 72h_2^{12}h_{-3}^3h_{-2}^8h_3^2 + 114h_2^5h_{-2}^7h_3^5h_{-3}^2 \\
& - 348h_2^{10}h_{-2}^6h_3^3h_{-3}^4 + 1092h_2^{12}h_{-3}^6h_3h_{-2}^2 - 312h_2^{13}h_{-3}^6h_3h_{-2}^2 \\
& + 72h_3^8h_{-2}^9h_{-3}h_2 - 216h_2^6h_3^5h_{-3}^6h_{-2}^2 + 108h_3^4h_{-2}^8h_2^6h_{-3} - 1130h_{-3}^5h_2^7h_3^4h_{-2}^3 \\
& - 317h_2^4h_{-3}^6h_3^3h_{-2}^6 + 144h_{-2}^{10}h_3^8h_{-3}h_2^2 - 780h_2^{11}h_{-3}^6h_3h_{-2} \\
& + 374h_2^5h_{-2}^7h_3^6h_{-3}^3 - 435h_3^7h_{-3}^4h_{-2}^5h_2^3 + 144h_{-2}^{10}h_3^7h_2^2 \\
& + 432h_{-2}^{13}h_{-3}^5h_{-2}^3 - 765h_2^{10}h_{-3}^7h_3^2 - 144h_3^7h_{-2}^{11}h_2^3 \\
& + 72h_{-3}^5h_{-2}^5h_2^{15} - 216h_3^9h_{-2}^8h_{-3}^2 - 324h_{-3}^5h_2^{14}h_{-2}^4 - 180h_2^{12}h_{-3}^5h_{-2}^2)p_1 = 0.
\end{aligned}$$

A-2. Section 4.3

$$\begin{aligned}
Eq_{6,1} : & (36 h_2^5 h_{-3}^2 + 36 h_2^2 h_3^2 h_{-2}^3) r_{2,1} \\
& + (-10 h_{-2}^3 h_3^3 + 6 h_2^3 h_3 h_{-2} h_{-4} + 6 h_3 h_4 h_2 h_{-2}^3 - 6 h_3 h_4 h_2^2 h_{-2}^4 - 3 h_2 h_{-2}^2 h_4^2 h_{-3} \\
& - 6 h_2^4 h_3 h_{-2}^2 h_{-4} + 10 h_3 h_2^4 h_{-3}^2 h_{-2} + 4 h_3^2 h_{-2}^2 h_4 h_{-3} + 4 h_2^2 h_{-3} h_3^2 h_{-4} - 3 h_2^3 h_{-3} h_{-4} h_4 \\
& - 10 h_3 h_2^3 h_{-3}^2 + 10 h_2 h_{-2}^4 h_3^3) p_1 = 0, \\
Eq_{6,2} : & (-8 h_2^3 h_3 h_{-2} h_{-3}^2 - 8 h_3^3 h_{-2}^4) r_{1,-2} + (12 h_2^5 h_{-3}^2 h_{-2} + 12 h_2^2 h_{-2}^4 h_3^2) r_{2,-1} \\
& + (h_2^3 h_{-3} h_{-2}^2 h_4 + 2 h_2^3 h_{-3} h_3^2 h_{-2}^3 - h_2^6 h_{-2} h_{-4} h_{-3} - h_2^4 h_{-3} h_3^3 h_4 + h_2^5 h_{-4} h_{-3} \\
& + 2 h_2^6 h_{-3}^3) p_1 = 0 \\
Eq_{6,3} : & (36 h_2^3 h_{-2}^4 h_{-3} h_3^2 + 36 h_2^6 h_{-2} h_{-3}^3) r_{2,-1} \\
& + (6 h_{-3}^4 h_2^7 - 3 h_2 h_{-2}^4 h_4^2 h_3 h_{-3} + 3 h_{-2}^5 h_4 h_2 h_3^2 - 3 h_{-2}^3 h_4 h_2^5 h_{-3}^2 + 4 h_{-2}^4 h_4 h_3^3 h_{-3} \\
& + 3 h_{-2}^2 h_4 h_2^4 h_{-3}^2 + 10 h_2^4 h_3^2 h_{-3}^2 h_{-2}^3 - 3 h_{-4} h_2^4 h_3^2 h_{-2}^4 - 10 h_3^4 h_{-2}^5 + 4 h_2^3 h_4 h_3^3 h_{-3} h_{-2} \\
& + 4 h_4^4 h_{-2}^6 h_2 - 10 h_{-2}^2 h_2^3 h_{-3}^2 h_3^2 - 3 h_2^3 h_{-3} h_{-2}^2 h_3 h_{-4} h_4 - 3 h_{-4} h_2^7 h_{-3}^2 h_{-2} \\
& + 3 h_2^6 h_{-4} h_{-3}^2 + 3 h_{-4} h_2^3 h_3^2 h_{-2}^3 - 3 h_{-2}^6 h_4 h_2^2 h_3^2) p_1 = 0, \\
Eq_7 : & (216 h_{-2} h_2^8 h_{-3}^4 + 432 h_2^3 h_{-2}^4 h_2^5 h_{-3}^2 + 216 h_3^4 h_{-2}^7 h_2^2) p_3 \\
& + (20 h_3^4 h_{-2}^6 h_2 + 40 h_3 h_2^7 h_{-3}^5 - 36 h_2^4 h_{-4} h_3 h_{-2}^3 h_4 h_{-3} + 18 h_2^2 h_3 h_{-2}^5 h_4^2 h_{-3} \\
& + 18 h_2^4 h_{-2}^2 h_4^2 h_{-3}^2 h_{-4} - 12 h_2^3 h_{-2}^4 h_4 h_3^3 h_{-2}^3 - 12 h_2 h_{-2}^5 h_4^2 h_{-4} h_3^2 - 60 h_{-3}^3 h_2^7 h_3 h_{-2} h_{-4} \\
& - 68 h_3^3 h_{-2}^3 h_{-4} h_2^3 h_{-3} + 12 h_3^2 h_{-2}^2 h_4 h_{-3}^2 h_2^3 h_{-4} + 32 h_3^2 h_{-2} h_2^3 h_4 h_{-3}^4 \\
& + 8 h_3^2 h_{-2}^4 h_4^2 h_{-3}^2 h_2 - 48 h_3^4 h_{-2}^2 h_2^2 h_{-3}^2 h_{-4} + 44 h_2^4 h_3^3 h_{-2}^4 h_{-4} h_{-3} + 54 h_3 h_{-2}^2 h_2^4 h_3^3 h_4 \\
& - 66 h_{-3}^3 h_2^5 h_{-2}^3 h_3 h_4 + 36 h_2^5 h_3 h_{-2}^4 h_4 h_{-4} h_{-3} - 18 h_4^2 h_3 h_{-2}^6 h_2^3 h_{-3} - 16 h_3^5 h_{-2}^6 h_2 h_{-3} \\
& + 24 h_3^3 h_{-2}^3 h_2^4 h_{-3}^3 + 38 h_2^2 h_3^3 h_{-2}^6 h_4 h_{-3} + 24 h_3 h_{-2} h_2^6 h_{-5} h_{-3}^2 + 54 h_2^7 h_3 h_{-2}^2 h_{-4}^2 h_{-3} \\
& + 16 h_3^4 h_{-2}^5 h_4 h_{-4} - 20 h_2^5 h_{-3}^2 h_3^2 h_{-4}^2 - 24 h_{-2} h_2^4 h_4^2 h_{-3}^4 - 16 h_3^4 h_{-2}^4 h_4 h_{-3}^2 \\
& - 12 h_3^2 h_{-2}^3 h_2^3 h_{-3} h_4 h_{-5} - 50 h_3^3 h_{-2}^5 h_4 h_{-3} h_2 - 24 h_3 h_{-2}^2 h_2^7 h_{-5} h_{-3}^2 + 24 h_{-2}^3 h_2^9 h_{-4} h_{-3}^2 \\
& + 16 h_2^2 h_3^4 h_{-2}^4 h_{-3}^2 + 15 h_2^6 h_{-3}^2 h_{-4}^2 h_4 + 40 h_3^3 h_{-2}^2 h_2^3 h_{-3}^3 + 16 h_3^2 h_2^5 h_{-3}^3 h_{-5} \\
& + 36 h_3 h_2^6 h_{-3}^3 h_{-4} + 24 h_3^3 h_{-2}^4 h_2^3 h_{-5} - 48 h_{-2}^2 h_2^8 h_{-4} h_{-3}^2 - 24 h_3^3 h_{-2}^5 h_2^4 h_{-5} \\
& - 54 h_3 h_{-2} h_2^6 h_{-4} h_{-3} + 48 h_3^2 h_{-2}^7 h_2^3 h_4 - 48 h_{-2}^4 h_2^6 h_4 h_{-3}^2 - 24 h_3^2 h_{-2}^6 h_2^6 h_{-4} \\
& + 48 h_3^2 h_{-2}^5 h_2^5 h_{-4} - 24 h_3^2 h_{-2}^8 h_2^4 h_4 + 52 h_3^2 h_{-2}^5 h_2^6 h_{-3}^2 - 32 h_3^2 h_{-2}^4 h_2^5 h_{-3}^2 \\
& + 24 h_{-2}^5 h_2^7 h_4 h_{-3}^2 - 12 h_2^6 h_{-3}^3 h_4 h_{-5} + 16 h_3^4 h_{-2}^3 h_2^2 h_{-3} h_{-5} - 24 h_{-2}^6 h_4 h_2^2 h_3^2 \\
& + 3 h_2^2 h_{-2}^4 h_4^3 h_{-3}^2 + 24 h_{-4} h_2^7 h_{-3}^2 h_{-2} + 24 h_{-2}^3 h_4 h_2^5 h_{-3}^2 - 20 h_2^4 h_3^2 h_{-3}^2 h_{-2}^3 \\
& - 24 h_{-4} h_2^4 h_3^2 h_{-2}^4 - 40 h_{-4}^2 h_2^7 - 16 h_2^9 h_{-3}^4 h_{-2}^2 + 40 h_3^5 h_{-2}^5 h_{-3} + 56 h_{-2} h_2^8 h_{-3}^4 \\
& + 68 h_3^4 h_{-2}^8 h_2^3 - 88 h_3^4 h_{-2}^7 h_2^2) p_1 = 0.
\end{aligned}$$

Step 8 in Section 4.3.2, formulae for $h_{\pm 5}$:

$$\begin{aligned}
h_5 = & -\frac{1}{12h_2^4h_{-3}^3h_{-2}(h_2^3h_{-3}^2+h_{-2}^3h_3^2)}(36h_{-3}^3h_2^5h_{-4}h_{-2}^2h_3^3+12h_2^4h_{-3}^2h_3^2h_{-2}^5h_4 \\
& -10h_3^6h_{-2}^8-30h_2^4h_{-3}^3h_{-2}^4h_4^2h_3-12h_2^7h_3^2h_{-2}^4h_{-4}h_{-3}^2 \\
& -30h_2^{10}h_{-3}^6-12h_2^5h_{-2}^6h_3^2h_4h_{-3}^2+20h_3^4h_{-2}^6h_2^4h_{-3}^2-14h_2^7h_3^2h_{-2}^3h_{-4}^4 \\
& -40h_2^6h_{-3}^4h_3^2h_{-2}^2-15h_2^7h_{-3}^4h_{-2}^2h_4+4h_3^5h_{-2}^7h_4h_{-3}+15h_2^{10}h_{-2}h_{-4}h_{-3}^4 \\
& +3h_2^3h_3^4h_{-2}^6h_{-4}+3h_2h_3^4h_{-2}^8h_4-50h_2^3h_{-3}^2h_3^4h_{-2}^5+15h_2^8h_{-3}^2h_4h_{-3}^4 \\
& -3h_2^2h_3^4h_{-2}^9h_4-3h_2^4h_3^4h_{-2}^7h_{-4}-15h_2^9h_{-4}h_{-3}^4+4h_2h_3^6h_{-2}^9 \\
& +12h_2^6h_{-3}^2h_3^2h_{-2}^3h_{-4}+20h_4h_{-3}^3h_2^3h_{-2}^4-20h_4h_{-3}^5h_2^6h_3h_{-2} \\
& -3h_2h_{-3}h_{-2}^7h_4^2h_3^3-30h_2^6h_{-3}^3h_{-2}^2h_{-4}h_4h_3+3h_2^5h_{-3}^3h_3^3h_{-4}^2h_{-3}), \\
h_{-5} = & \frac{1}{12h_2^2h_3h_{-2}(h_2^3h_{-3}^2+h_{-2}^3h_3^2)}(-12h_2^2h_{-2}^7h_4h_3^2+30h_{-3}h_2^3h_3h_{-2}^3h_{-4}h_4 \\
& +12h_2^3h_{-2}^4h_{-4}h_3^2+50h_{-3}^2h_2^3h_3^2h_{-2}^3-4h_3^3h_{-2}^5h_4h_{-3}-40h_3h_{-2}^2h_{-3}^3h_4h_3^2 \\
& +10h_3^4h_{-2}^6-12h_2^4h_{-2}^5h_{-4}h_3^2-12h_2^4h_{-3}^3h_4h_{-3}^2+3h_2h_{-2}^5h_4^2h_3h_{-3}+10h_{-3}^2h_2^4h_{-2}^4h_3^2 \\
& +27h_{-3}h_2^5h_3h_{-2}h_{-4}^2+40h_2^6h_{-4}^3+12h_2^5h_{-4}^2h_4h_{-3}^2-12h_2^6h_{-2}h_{-4}h_{-3}^2-16h_{-3}^3h_2^5h_3h_{-4} \\
& +12h_2h_{-2}^6h_4h_3^2+12h_2^7h_{-2}^2h_{-4}h_{-3}^2-16h_{-3}^4h_2^7h_{-2}+26h_3^4h_{-2}^7h_2+20h_{-3}h_2^2h_3^3h_{-4}h_{-2}^3).
\end{aligned}$$

Note that $h_{-5} \neq \overline{h_5}$ in this formula. $h_{-5} - \overline{h_5} = 0$ gives an integrability equation which is quadratic in h_4, h_{-4} .

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